Tiling for Unique Factorization Domains

Brian Wichmann

July 7, 2013

1 Introduction

The building in which I worked for over 20 years had an interesting tiling pattern in the entrance hall. This tiling is shown in Figure 1. The result is shown on the front cover of a book on the history of NPL written by David Yates [4]. The book contains the following information about the pattern:

Floor tiles and the complex plane

> Donald Davies's design for the tiling of the entrance hall floor in the new building is of interst to mathematicians. Complex number of the form $a + b\omega$ (where a and b are integers and ω is one of the complex cube roots of unity $e^{2\pi i/3}$) form a **ring**, that is, if any two are added, subtracted or multiplied you get another member of the set. Some members of this ring are prime in the sense that they cannot be expressed as the product of other members except in trivial ways. The hall floor represents the complex plane, with the origin at the centre of the metal tile and the real axis pointing north. The centres of the hexagonal tiles are the members of the ring, primes correspond to dark tiles and non-primes to light. The pattern can be seen as 12 sectors, each a reflection of its neighbour in their common boundary. It has been used as the basis of the cover design on this book.

Wikipedia calls it the Eisenstein integers [1]. Also for a picture of the prime distribution, see Guy [2, Page 35], who calls them the Eisenstein-Jacobi integers. In the Figure, the single zero is shown in dark grey, the six units in a light grey and the primes in a middle grey. (The other hexagons are the composite values of the Unique Factorization Domain.)

The obvious question to ask is if similar patterns can be produced, and if so, how many. This then becomes a question of extensions of the integers into the complex plane in which the result is a Unique Factorization Domain (UFD). The answer to this question is that each UFD consist of the numbers $a + b\sqrt{-d}$ where d is one of the nine values 1, 2, 3, 7, 11, 19, 43, 67 and 163. In the case where $d > 2$, the values a and b are either integers or both $x + \frac{1}{2}$ (for integer x).

For a statement of this problem and its solution, see MathWorld [3].

2 Implementing the patterns

By choosing the imaginary axis to have a unit size of $\sqrt{d}$, the cases of $d = 1$ or 2 has a square lattice, while the remaining cases have a hexagonal lattice. This implies, for simplicity, that two programs are needed. Rather than use the circular plot that Guy uses, we produce an A4 result with the origin near the left bottom corner, since in all cases we have reflective symmetry about both axes.

The program to compute the primes and then generate the PostScript is written in Python. It is rather slow, not so much because of Python, but due to any degree of optimization in my coding. It should be straightforward only the perform the calculation in one quadrant, but this has not been done to be more confident of the correctness of the result.

Data needs to be produced for the tiling system's SQL database. This then ensures that the tiling patterns can be integrated with the rest of the system.

The Guassian integers are given in Figure 2. The other square lattice UFD is shown in Figure 3.

The hexagonal lattice UFDs start with the Eisenstein-Jacobi integers shown in Figure 4. The other hexagonal lattice UFDs (produced by scaling the y-axis appropriately) are shown in Figures 5, 6, 7, 8, 9 and 10.

Note that the only two UFDs based upon roots of unity are those of cube roots which inspired all of these tilings and the Guassian integers.

References

[1] Eisenstein Integer. Wikipedia.
http://en.wikipedia.org/wiki/Eisenstein_integer

[2] Richard K. Guy. Unsolved Problems in Number Theory. Springer. 2nd Edition. ISBN 0-387-94289-0. 1994.

[3] Gauss's Class Number Problem.
http://mathworld.wolfram.com/GausssClassNumberProblem.html

[4] D. M. Yates, Turing's Legacy. Science Museum. 1997. ISBN 09018-05947

[5] B. A. Wichmann, The World of Patterns, CD and booklet. World Scientific. 2001. ISBN 981-02-4619-6
http://www.worldscibooks.com/mathematics/4698.html

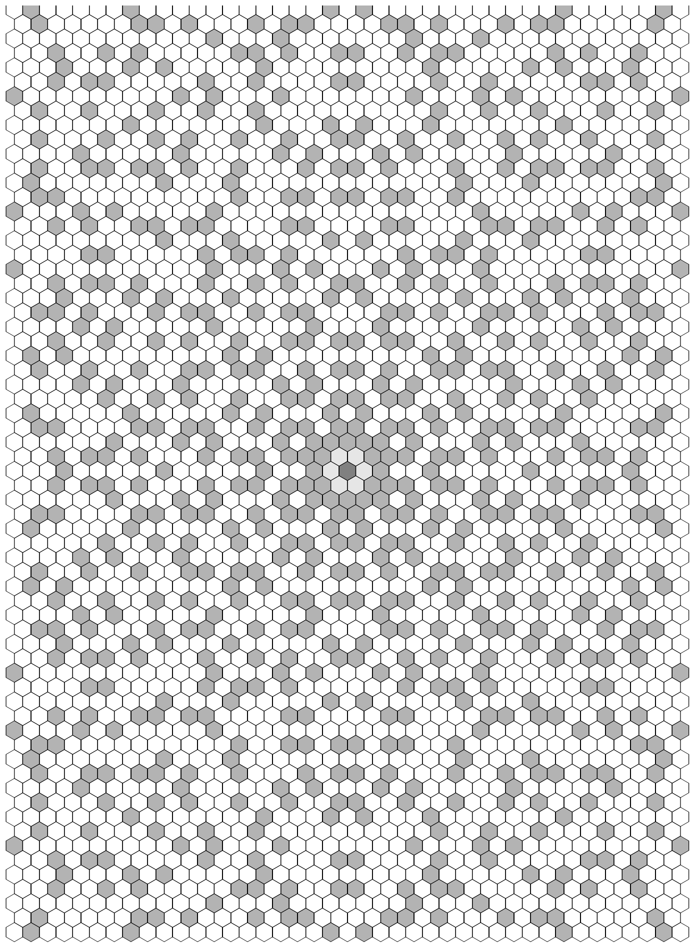

Figure 1: NPL, Building 93 hall floor pattern

This tiling pattern was designed by D. W. Davies, FRS one of the architects of the Internet.
This graphic was produced by Robin Barker of NPL by finding the primes using a program in ML, and then producing the graphic itself in PostScript.

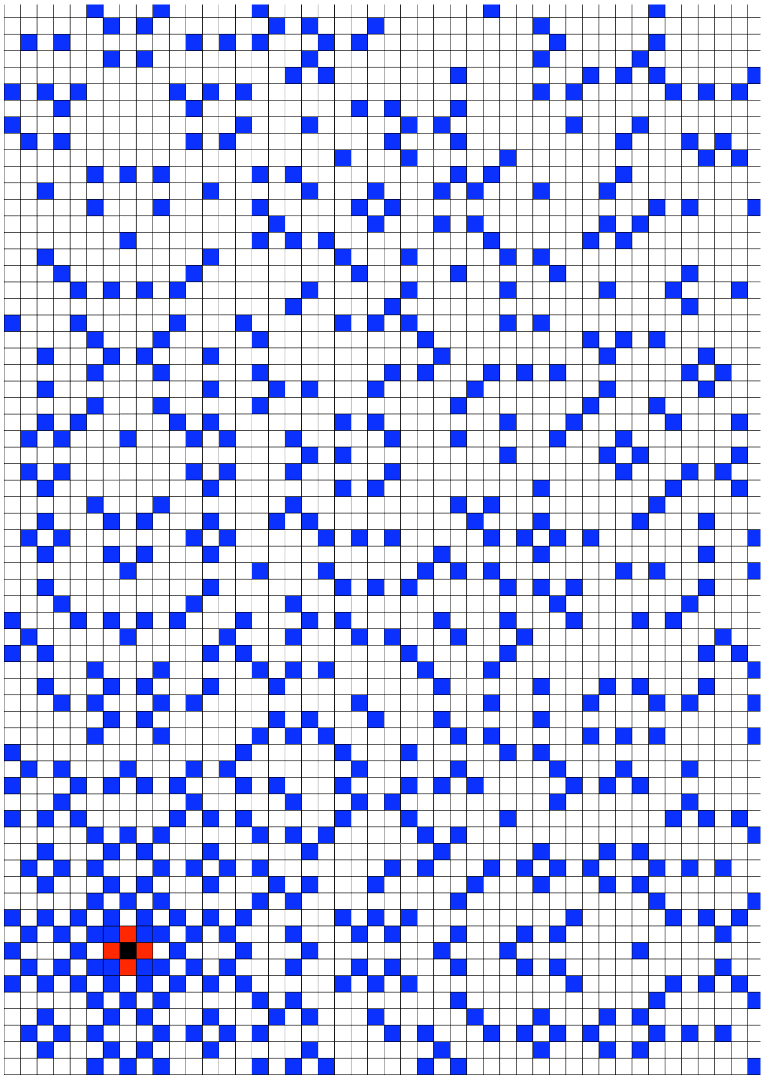

Figure 2: Guassian integers

The zero is black, the units red and the primes blue.

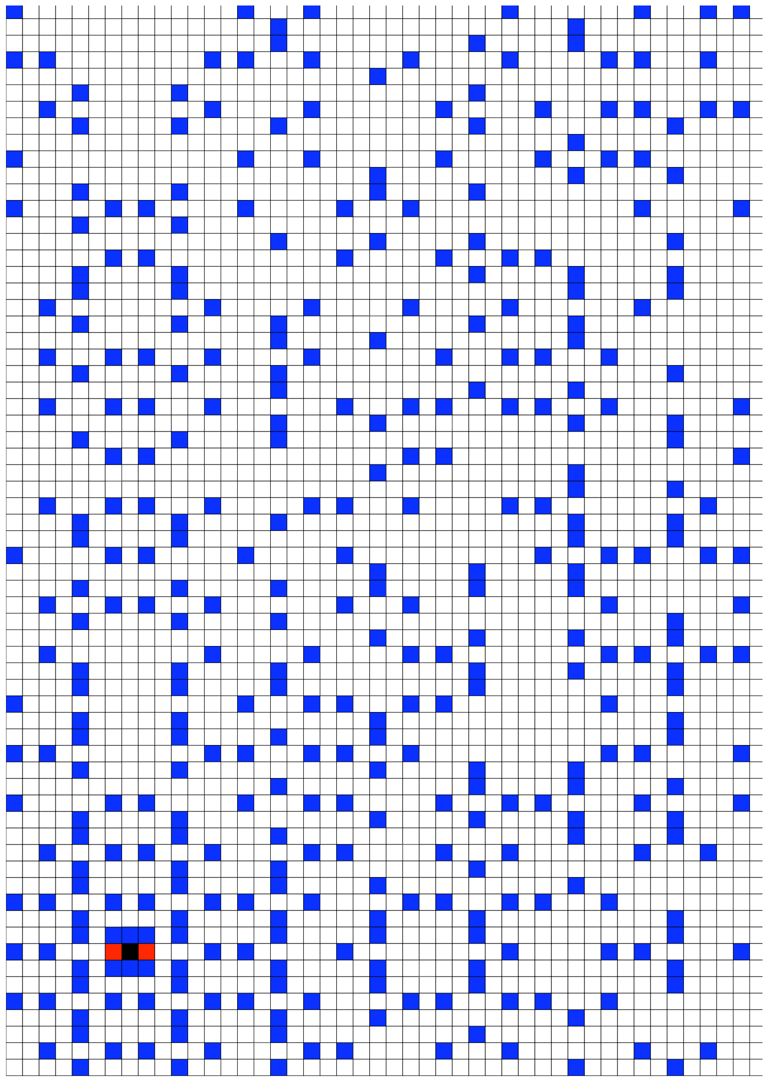

Figure 3: UFD: $a + b\sqrt{-2}$

The zero is black, the units red and the primes blue.

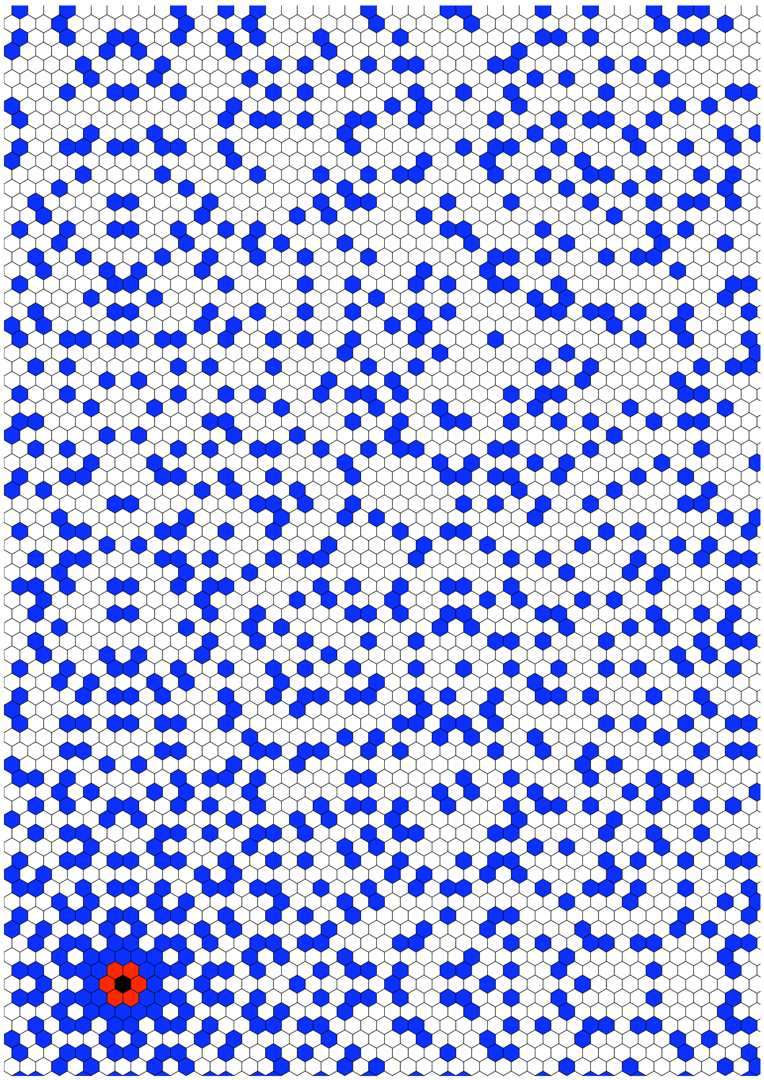

Figure 4: Eisenstein-Jacobi integers: $a + b\omega$

The zero is black, the units red and the primes blue.

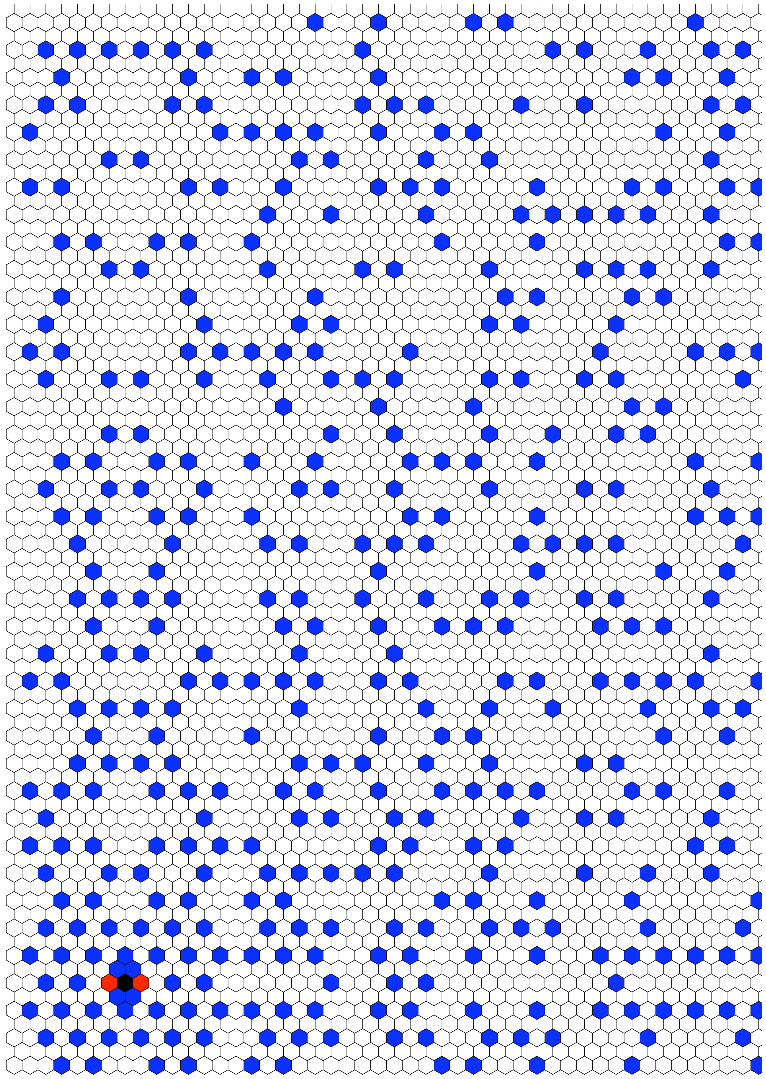

Figure 5: UFD: $a + b\sqrt{-7}$

The zero is black, the units red and the primes blue.

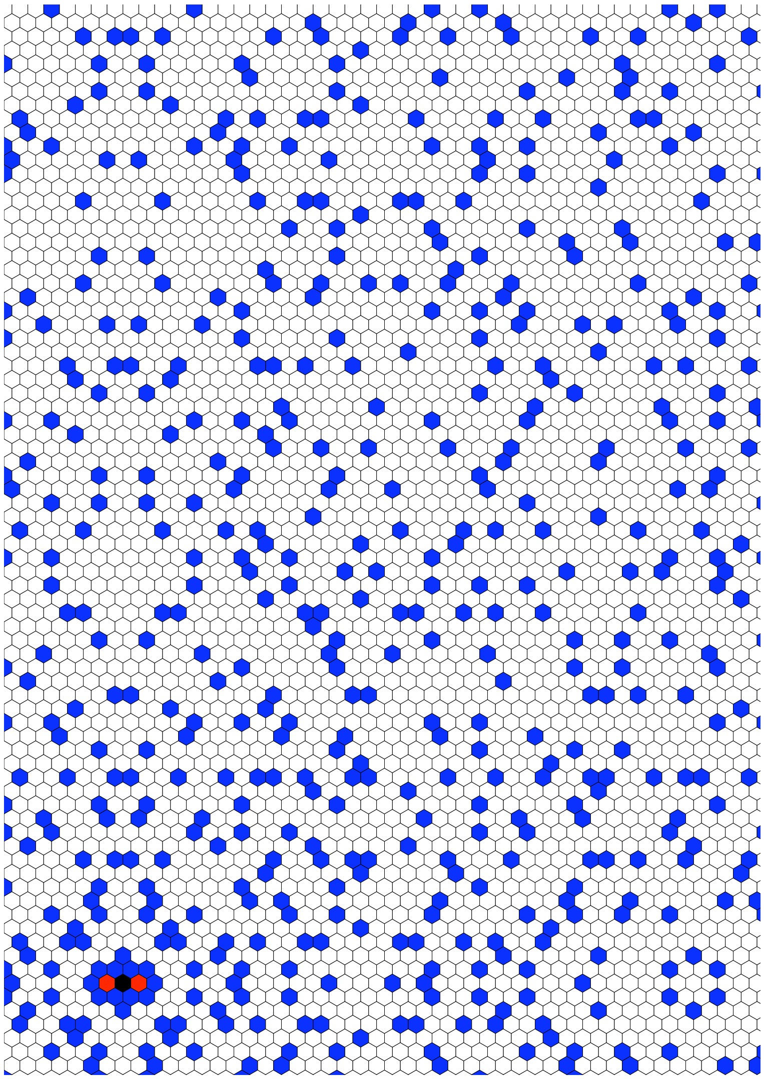

Figure 6: UFD: $a + b\sqrt{-11}$

The zero is black, the units red and the primes blue.

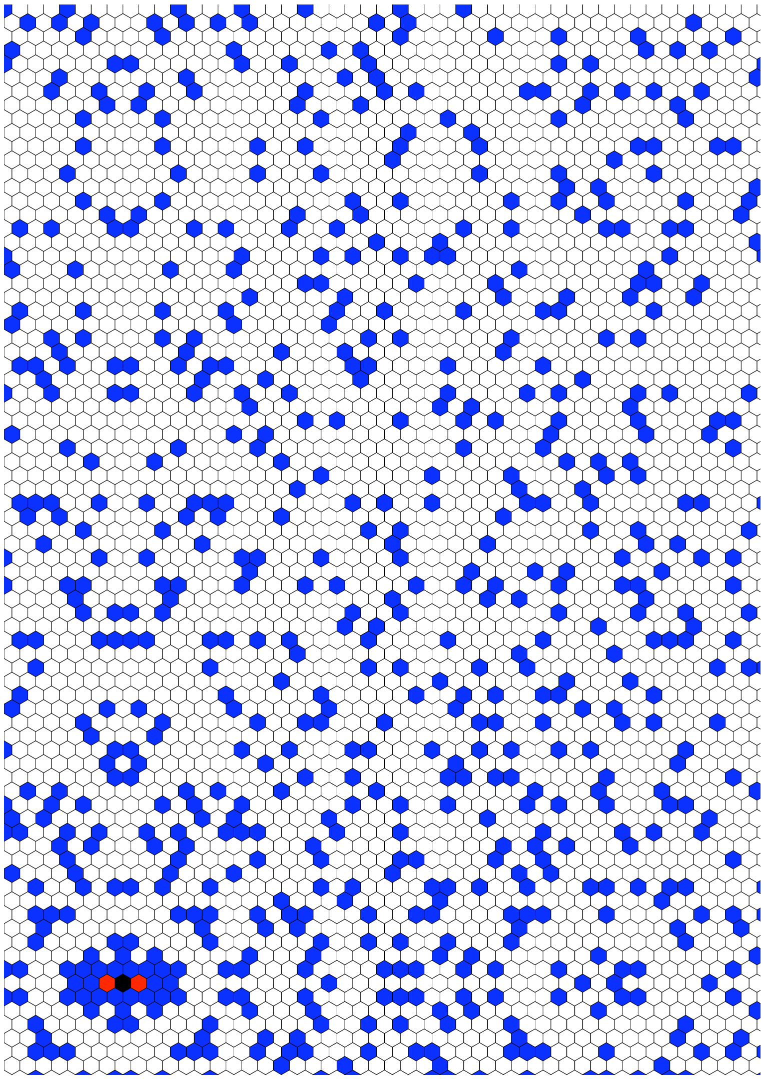

Figure 7: UFD: $a + b\sqrt{-19}$

The zero is black, the units red and the primes blue.

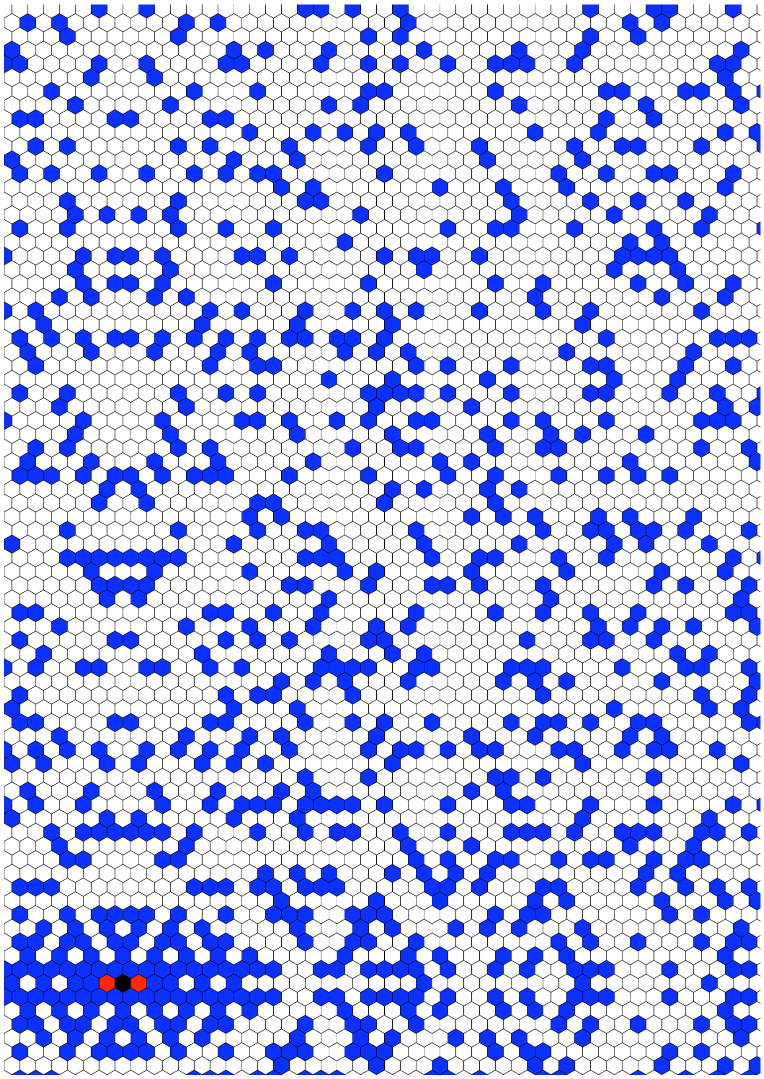

Figure 8: UFD: $a + b\sqrt{-43}$

The zero is black, the units red and the primes blue.

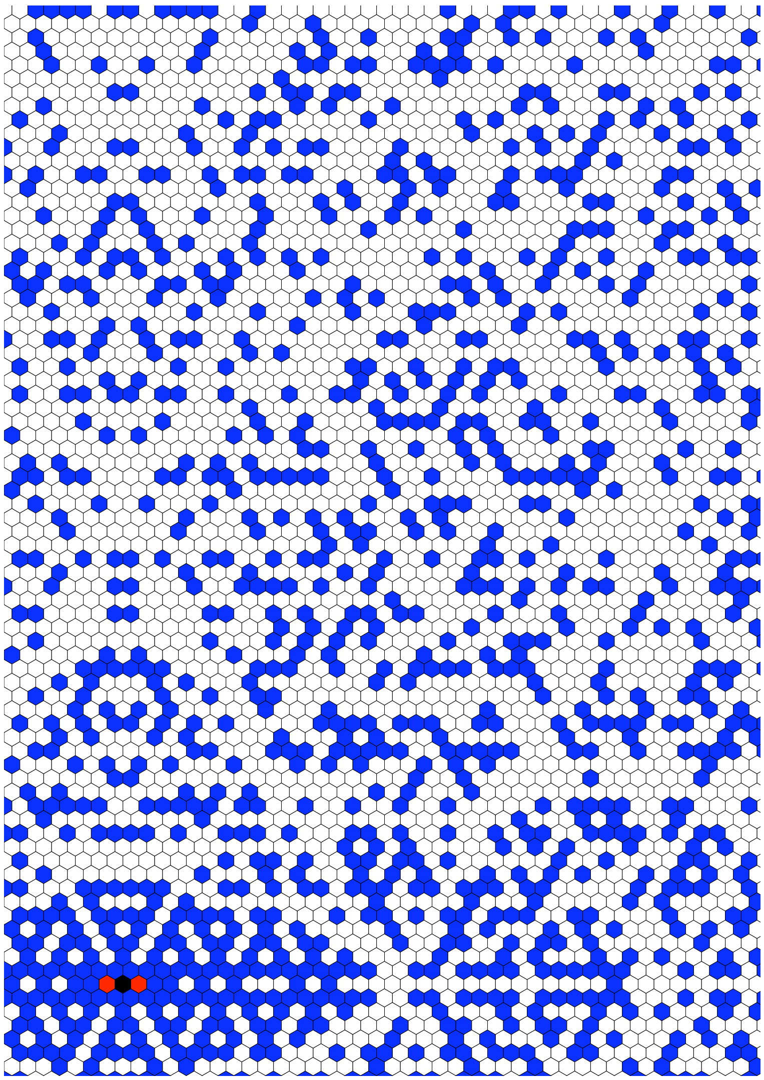

Figure 9: UFD: $a + b\sqrt{-67}$

The zero is black, the units red and the primes blue.

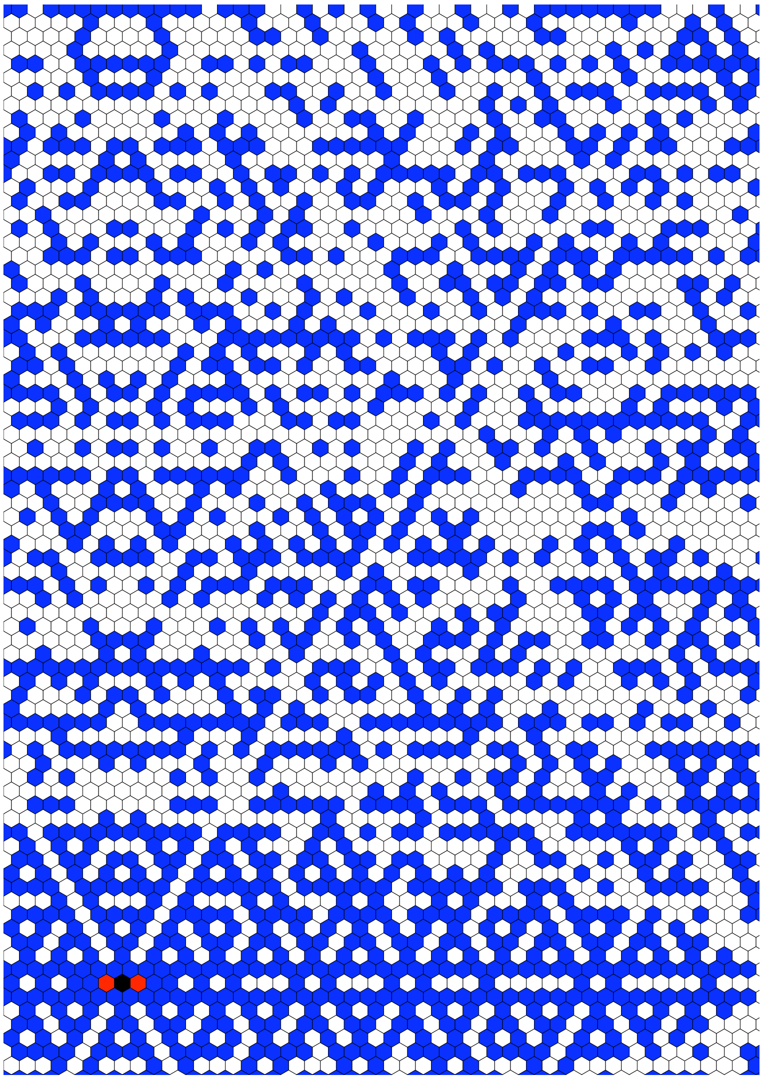

Figure 10: UFD: $a + b\sqrt{-163}$

The zero is black, the units red and the primes blue.

3 Revision record

1. Started, 31st March 2009.
2. First draft finished 9th April 2009.
3. Small revision, 16th April 2009.
4. Revised to take into account the comments by David Yates, 5th May 2009.
5. Revised to add links to tilingsearch.org, 7th July 2013.